

# A BIRMAN-SERIES TYPE RESULT FOR GEODESICS WITH INFINITELY MANY SELF-INTERSECTIONS

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**ABSTRACT.** Given a hyperbolic surface  $\mathcal{S}$ , a classic result of Birman and Series states that for each  $K$ , all complete geodesics with at most  $K$  self-intersections can only pass through a certain nowhere dense, Hausdorff dimension 1 subset of  $\mathcal{S}$ . We define a self-intersection function for each complete geodesic, which bounds the number of self-intersections in finite length subarcs. We then extend the Birman-Series result to sets of complete geodesics with certain bounds on their self-intersection functions. In fact, we get the same conclusion as the Birman-Series result for sets of complete geodesics whose self-intersection functions are in  $o(l^2)$ , where  $l$  measures arclength.

## 1. INTRODUCTION

Let  $\mathcal{S}$  be a genus  $g$  surface with  $n$  boundary components, and let  $X$  be a hyperbolic metric on  $\mathcal{S}$  in which each boundary component is geodesic. Consider the set  $\mathcal{G}$  of complete geodesics on  $\mathcal{S}$ . In particular, geodesics in  $\mathcal{G}$  never hit the boundary of  $\mathcal{S}$ .

Given any subset  $\mathcal{H} \subset \mathcal{G}$ , we define the **image** of  $\mathcal{H}$  in  $\mathcal{S}$ , denoted  $\mathbf{Im} \mathcal{H}$ , to be the set of points in  $\mathcal{S}$  that lie on some curve in  $\mathcal{H}$ . Birman and Series showed that, for each  $K$ , if  $\mathcal{H}$  is the set of all complete geodesics with at most  $K$  self-intersections, then  $\mathbf{Im} \mathcal{H}$  is nowhere dense and has Hausdorff dimension 1 [BS85].

In this paper, we find much weaker conditions on subsets  $\mathcal{H} \subset \mathcal{G}$  that give the same conclusion. We get these conditions by studying the self-intersection function of each  $\gamma \in \mathcal{G}$ , which is defined as follows. Take a complete geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{S}$  parameterized by arclength. Let  $\gamma_l = \gamma|_{[-\frac{l}{2}, \frac{l}{2}]}$  be the length  $l$  subarc of  $\gamma$  centered at  $\gamma(0)$ . Then  $f(l) = i(\gamma_l, \gamma_l)$  is the **self-intersection function** of  $\gamma$ .

This function depends on the parameterization of  $\gamma$ , so we choose a parameterization for each  $\gamma \in \mathcal{G}$  so that its self-intersection function is as small as possible. That is, if  $\gamma : \mathbb{R} \rightarrow \mathcal{S}$  and  $\gamma' : \mathbb{R} \rightarrow \mathcal{S}$  are two parameterizations by arclength of the same complete geodesic, then  $\gamma$  has a smaller self-intersection function than  $\gamma'$  if  $i(\gamma_l, \gamma_l) \lesssim i(\gamma'_l, \gamma'_l)$ . Note that we write  $A(l) \lesssim B(l)$  if  $\limsup_{l \rightarrow \infty} \frac{A(l)}{B(l)} \leq 1$ . There need not be a parameterization of  $\gamma$  with the least self-intersection function, so we make an arbitrary choice of parameterization for each  $\gamma \in \mathcal{G}$ .

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is any function. Let

$$\mathcal{G}(f) = \{\gamma \in \mathcal{G} \mid i(\gamma_l, \gamma_l) \lesssim f(l)\}$$

**Theorem 1.1.** *For any  $k > 0$ , suppose  $f(l)$  is a function with  $f(l) \leq (kl)^2$  for all  $l$  large enough. Then the Hausdorff dimension of  $\mathbf{Im} \mathcal{G}(f)$  is at most  $\mu(k)$ , where  $\lim_{k \rightarrow 0} \mu(k) = 1$ . In particular, if  $f(l) = o(l^2)$ , then  $\mathbf{Im} \mathcal{G}(f)$  has Hausdorff dimension 1.*

On the other hand, [LS15] implies that  $\mathbf{Im} \mathcal{G}(f)$  is dense whenever  $f(l)$  is super-linear in  $l$  (see Section 1.2). Nevertheless, we get a nowhere density result once we get more control on the self-intersection function of our geodesics. So if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is any function, then let

$$\mathcal{G}(f, L) = \{\gamma \in \mathcal{G} \mid i(\alpha, \alpha) \leq f(l), \forall \alpha \subset \gamma \text{ s.t. } l(\alpha) \geq L\}$$

be the set of  $\gamma \in \mathcal{G}$  such that all length  $l$  subarcs have at most  $f(l)$  self-intersections, whenever  $l \geq L$ .

**Theorem 1.2.** *There is a  $k_0 > 0$  so that if  $f(l) \leq (k_0 l)^2$  for all  $l$ , then  $\mathbf{Im} \mathcal{G}(f, L)$  is nowhere dense for all  $L \geq 0$ .*

**Remark 1.3.** *Both the original result of Birman and Series, as well as Theorems 1.1 and 1.2, still hold when  $\mathcal{S}$  has a negatively curved metric with curvature bounded away from zero and infinity. However, the function  $\mu$  and constant  $k_0$  will depend on the metric. This is because the results below that use hyperbolic geometry can be proven in the general negative curvature case, but with different constants.*

**1.1. Previous results for complete geodesics.** Complete geodesics on  $\mathcal{S}$  satisfy the following dichotomy. On the one hand, when  $X$  is a complete, finite volume metric, then  $\mathbf{Im} \mathcal{G} = \mathcal{S}$ . Even when  $X$  has geodesic boundary,  $\mathbf{Im} \mathcal{G}$  can have Hausdorff dimension greater than 1, and points of Lebesgue density. In the case of a pair of pants, this is proven in [HJL12].

Moreover, when  $X$  has finite volume, any “typical” geodesic in  $\mathcal{G}$  has dense image in  $\mathcal{S}$  in the following sense. Let  $T_1 \mathcal{S}$  be the unit tangent bundle of  $\mathcal{S}$ . Then we can choose a vector  $v \in T_1 \mathcal{S}$  at random with respect to Lebesgue measure, and consider the complete geodesic  $\gamma$  with tangent vector  $\gamma'(0) = v$ . By mixing of the geodesic flow,  $\mathbf{Im} \gamma$  will be dense with probability 1. Note that in this case, mixing of the geodesic flow also implies that the self-intersection function of  $\gamma$  will grow asymptotically like  $\kappa l^2$ , with probability 1.

On the other hand, the classical result of Birman and Series that we reference above shows that when  $f(l) = K$  is constant, meaning that  $\mathcal{G}(f)$  consists of complete geodesics with at most  $K$  self-intersections, then  $\mathbf{Im} \mathcal{G}(f)$  has Hausdorff dimension 1 and is nowhere dense [BS85].

So we can think of self-intersection functions on a sliding scale. On one end of the scale, we have functions with  $f(l) = O(1)$ , and on the other side, we have functions with  $f(l) = O(l^2)$ . Theorems 1.1 and 1.2 allow us to interpolate between these two extremes. They indicate that the transition from Hausdorff dimension 1 to Hausdorff dimension 2, and from being nowhere dense to being dense, occur at the far end of the scale, among functions with  $f(l) = O(l^2)$ .

**1.2. Contrast with results for closed geodesics.** There is an analogous story for closed geodesics. Let  $\mathcal{G}^c$  be the set of closed geodesics on  $\mathcal{S}$ . When  $X$  has finite volume (and no boundary),  $\mathbf{Im} \mathcal{G}^c$  is dense in  $\mathcal{S}$  by the closing lemma and mixing of the geodesic flow. On the other hand, the set of simple closed geodesics has nowhere dense image by [BS85].

Recently, Lenzhen and Souto have considered the sets of closed geodesics between these two extremes [LS15]. In particular, for any function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , they consider the set

$$\{\gamma \in \mathcal{G}^c \mid i(\gamma, \gamma) \leq f(l(\gamma))\}$$

They show that the image of this set is dense whenever  $\lim_{l \rightarrow \infty} f(l)/l = \infty$ , and that its lift to  $T_1\mathcal{S}$  has Hausdorff dimension strictly smaller than 3 if  $f(l) = o(l)$ .

The first part of their result can be combined with our theorems to get the following corollary.

**Corollary 1.4** (Consequence of [LS15] and Theorem 1.1). *If  $\lim_{l \rightarrow \infty} f(l)/l = \infty$ , then  $\mathbf{Im} \mathcal{G}(f)$  is dense. In particular, there is some  $k_0 > 0$  so that for any  $k < k_0$ ,  $\mathbf{Im} \mathcal{G}(k^2 l^2)$  is dense, but does not have full Hausdorff dimension.*

*Proof.* Suppose  $\lim_{l \rightarrow \infty} f(l)/l = \infty$ . This corollary follows from the fact that if  $\gamma$  is a closed geodesic with  $l(\gamma) = L$  and  $i(\gamma, \gamma) \leq f(L)$ , then  $\gamma \in \mathcal{G}(f)$ .

To see this, note that we can view any closed geodesic  $\gamma$  as a complete geodesic  $\gamma : \mathbb{R} \rightarrow \mathcal{S}$  parameterized by arclength. If  $l(\gamma) = L$ , then this parameterization has period  $L$ . We can define  $\gamma_l$  as above to be the length  $l$  subarc of  $\gamma$  centered at  $\gamma(0)$ , where  $\gamma_l$  is defined for any  $l \in \mathbb{R}$ . Then

$$\frac{i(\gamma_l, \gamma_l)}{l} \lesssim \frac{f(L)}{L}$$

Because  $\lim_{l \rightarrow \infty} f(l)/l = \infty$ , it is trivially true that  $\lim_{l \rightarrow \infty} \frac{f(L)}{L} \cdot \frac{l}{f(l)} = 0$ , since  $L$  is a constant. Therefore  $i(\gamma_l, \gamma_l) \lesssim f(l)$ . In other words,  $\gamma \in \mathcal{G}(f)$ .

Thus,  $\mathbf{Im} \mathcal{G}(f)$  contains a dense set by [LS15], and so it is dense.  $\square$

On the other hand, a closed geodesic  $\gamma$  with  $l(\gamma) = L$  and  $i(\gamma, \gamma) \leq f(L)$  does not necessarily belong to  $\mathcal{G}(f, L')$ , if  $L' < L$ . In fact, to determine whether  $\gamma \in \mathcal{G}(f, L')$ , one would have to examine how the self-intersections of  $\gamma$  are distributed along its length. So [LS15] does not contradict Theorem 1.2.

It is interesting to note that the transition away from full Hausdorff dimension occurs when  $f(l) = (k_0 l)^2$  for complete geodesics, while it occurs around  $f(l) = O(l)$  for closures of sets of closed geodesics.

**1.3. Reduction to closed surfaces.** It is enough to prove Theorems 1.1 and 1.2 for closed surfaces  $\mathcal{S}$  (without boundary). In fact, if  $X$  has geodesic boundary, then we can double  $\mathcal{S}$  across this boundary to get a  $\mathcal{S}'$  with finite volume metric  $X'$ . There is a natural inclusion  $\mathcal{S} \hookrightarrow \mathcal{S}'$  along which  $X'$  pulls back to  $X$ . Any  $\gamma \subset \mathcal{S}$  gets sent to a geodesic on  $\mathcal{S}'$  with the same self-intersection function. So if  $\mathbf{Im} \mathcal{G}(f)$  has Hausdorff dimension  $h$ , or is nowhere dense, on  $\mathcal{S}'$ , then the same is true on  $\mathcal{S}$ .

**1.4. Structure of the paper.** In Section 2, we show how to approximate any complete geodesic  $\gamma$  by a sequence of closed geodesics  $\gamma_n$  that also approximate the self-intersection function of  $\gamma$ . In particular, given any geodesic arc  $\alpha$  of length  $L$ , which can be thought of as a subarc of  $\gamma$ , we show how to find a nearby closed geodesic whose length and self-intersection number are not much larger than those of  $\alpha$  (Lemma 2.2).

In Section 3, we apply Lemma 2.2 to construct covers for  $\mathcal{G}(f)$  and  $\mathcal{G}(f, L)$  that consist of regular neighborhoods of closed geodesics. In particular, for each function  $f$ , we get a sequence of finite covers  $\{\mathcal{C}_n\}$ . In Lemma 3.1, we show that  $\mathcal{C}_n$  covers  $\mathcal{G}(f, L)$  for all  $n$  large enough (depending on  $L$ .) Moreover, we show that any infinite union of these covers is, in fact, a cover for  $\mathcal{G}(f)$  (Lemma 3.2).

In Section 4, we approximate the number of open sets in the cover  $\mathcal{C}_n$ , for each  $n$ . The set  $\mathcal{C}_n$  is a collection of regular neighborhoods of closed geodesics that lie

in a certain set. So to approximate the size of  $\mathcal{C}_n$ , we need to approximate the size of this set of closed geodesics. We do this in Lemma 4.1.

In Sections 5 and 6, we prove Theorems 1.2 and 1.1, respectively, given the above lemmas. We do this by getting upper bounds the Lebesgue and Hausdorff measures of each cover  $\mathcal{C}_n$ , where the measure of a cover is defined to be the measure of the union of elements of that cover.

**1.5. Notation.** There are several points in this paper where we only need coarse estimates. We use the following notation. If two functions  $A(x)$  and  $B(x)$  satisfy  $A(x) \leq cB(x)$  where  $c$  is a constant depending only on some quantity  $D$ , then we write

$$A(x) \preccurlyeq B(x)$$

and say that the constants depend only on  $D$ . We will also say that  $A(x)$  is coarsely bounded by  $B(x)$ .

Furthermore, given two curves  $\alpha$  and  $\beta$ ,  $i(\alpha, \beta)$  denotes the least number of self-intersections between all curves freely homotopic to  $\alpha$  and  $\beta$ . On the other hand,  $\#\alpha \cap \beta$  denotes the number of transverse intersections between the curves  $\alpha$  and  $\beta$  themselves. In particular, we use the notation  $\#\alpha \cap \beta$  when  $\alpha$  and  $\beta$  are arcs rather than closed curves.

## 2. LEMMAS ABOUT ARCS

It is well-known that one can approximate any complete geodesic with a sequence of closed geodesics. For example, closed geodesics are dense in the space of geodesic currents, which also contain the set of complete geodesics [Bon88]. So given any complete geodesic  $\gamma_\infty$ , we can find a sequence  $\{\gamma_i\} \subset \mathcal{G}^c$  so that  $\lim \gamma_i = \gamma_\infty$ . Note that this limit holds in the measure-theoretic sense of geodesic currents, but it must also hold as a Hausdorff limit of the geodesics themselves. See [Bon88] for more details.

Suppose  $\gamma_\infty \in \mathcal{G}(f)$ . Then not only do we want to approximate  $\gamma_\infty$  by a sequence  $\{\gamma_i\}$  of closed geodesics, we want the self-intersection numbers of the closed geodesics to eventually be coarsely bounded by the self-intersection function of  $\gamma_\infty$ . That is, if  $l(\gamma_i) = l_i$ , we want  $i(\gamma_i, \gamma_i) \preccurlyeq f(l_i)$ , where the constant is independent of  $i$ .

We make this precise in terms of subarcs of complete geodesics. In particular, given a point  $x \in \mathbf{Im} \gamma_\infty$ , we can take a nested sequence of subarcs  $\alpha_l \subset \gamma_\infty$  centered at  $x$  so that  $\alpha_l$  has length  $l$ . Then the following lemma says we can find a closed geodesic close to  $\alpha_l$  that does not have too much more length or too many more self-intersections.

**Definition 2.1.** We say a closed geodesic  $\gamma$   **$r$ -fellow travels** a geodesic arc  $\alpha$  if there are some lifts  $\tilde{\gamma}$  and  $\tilde{\alpha}$  of  $\gamma$  and  $\alpha$ , respectively, to the universal cover  $\tilde{\mathcal{S}}$  of  $\mathcal{S}$  so that  $\tilde{\alpha}$  lies in a  $r$ -neighborhood of  $\tilde{\gamma}$ .

Let

$$\mathcal{G}_L^c(K) = \{\gamma \in \mathcal{G}^c \mid l(\gamma) \leq L, i(\gamma, \gamma) \leq K\}$$

be the set of closed geodesics with length at most  $L$  and with at most  $K$  self-intersections.

**Lemma 2.2.** *There is a constant  $d$  depending only on the metric  $X$  so that the following holds. Let  $\alpha$  be a geodesic arc of length  $L \geq d$  with  $\#\alpha \cap \alpha = K$ . Then there exists a closed geodesic*

$$\gamma \in \mathcal{G}_{2L}^c(K + dL)$$

that 1-fellow-travels  $\alpha$ .

This Lemma is a direct consequence of Claims 2.4 and 2.3 below. The proof of Lemma 2.2 given these claims is at the end of this section.

**Claim 2.3.** *For any geodesic arc  $\alpha$  with  $l(\alpha) \geq 3$  there is a  $\gamma \in \mathcal{G}^c$  so that  $\gamma$  1-fellow travels  $\alpha$  and  $l(\gamma) \leq l(\alpha) + R$ , where  $R$  is a constant depending only on  $X$ .*

We would like to thank Chris Leininger for suggesting the idea for this claim and its proof.

*Proof.* Suppose  $\beta$  is a geodesic arc so that  $\alpha$  and  $\beta$  can be concatenated into a closed curve  $\gamma'$ . Then  $\gamma'$  is a piecewise geodesic closed curve with corners at the endpoints of  $\alpha$ . Suppose the angle deficit at each corner is at most  $\epsilon$  (Figure 1). Let  $\gamma$  be the geodesic representative of  $\gamma'$ . Then  $\gamma$  must  $D(\epsilon)$ -fellow travel  $\gamma'$ , for

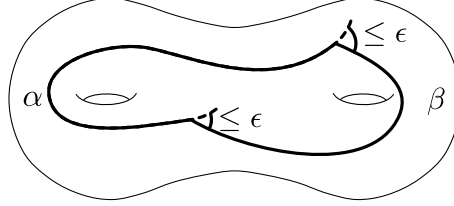


FIGURE 1.

a function  $D(\epsilon)$  depending only on the maximal angle deficit  $\epsilon$  with

$$\lim_{\epsilon \rightarrow 0} D(\epsilon) = 0$$

To see this, we will round the corners of  $\gamma'$  to get a nearby, piecewise  $C^2$  curve  $\gamma_c$  that  $\epsilon \sinh(1)$ -fellow travels  $\gamma$ . The curve  $\gamma_c$  will have geodesic curvature bounded by a function  $g(\epsilon)$  at each point, where  $\lim_{\epsilon \rightarrow 0} g(\epsilon) = 0$ . So by [Lei06], we can conclude that  $\gamma$  must  $f(\epsilon)$ -fellow travel  $\gamma_c$ , where  $f(\epsilon)$  is a continuous function in  $\epsilon$ , with  $f(0) = 0$ .

First, if  $l(\beta) \leq 2$ , we need to modify  $\gamma'$  slightly: Replace  $\gamma'$  by the curve  $\gamma''$  that is freely homotopic to it relative one of the endpoints of  $\alpha$ . There are lifts  $\tilde{\gamma}'$  and  $\tilde{\gamma}''$  to the universal cover  $\tilde{\mathcal{S}}$  of  $\mathcal{S}$  that form a geodesic triangle  $\triangle abc$  where  $c$  is the vertex opposite  $\tilde{\gamma}''$  and has angle at least  $\pi - \epsilon$  (Figure 2). Applying the hyperbolic law of sines, we see that the distance from  $\tilde{\gamma}'$  to  $\tilde{\gamma}''$  is at most  $\epsilon \sinh(2)$ .

The rest of the proof is essentially the same, whether we deal with  $\gamma'$  or  $\gamma''$ . So assume for what follows that  $l(\beta) > 2$ , so we deal with  $\gamma'$ . Take a bi-infinite lift  $\tilde{\gamma}'$  of  $\gamma'$  to  $\tilde{\mathcal{S}}$ .

The curve  $\tilde{\gamma}'$  is a piecewise geodesic with angle deficit at most  $\epsilon$  at its corners. We will now find a nearby piecewise  $C^2$  curve  $\tilde{\gamma}_c$  whose curvature is bounded above by  $g(\epsilon)$  at each point, where  $\lim_{\epsilon \rightarrow 0} g(\epsilon) = 0$ .

For this, we use the upper half plane model for  $\mathbb{H}^2$ . Since  $X$  is a hyperbolic metric, we can view  $\tilde{\mathcal{S}}$  as a subset of  $\mathbb{H}^2$ . Applying a hyperbolic isometry, we can

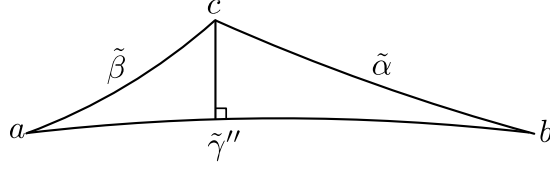
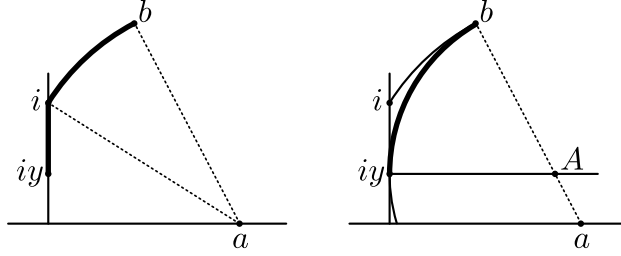


FIGURE 2.

assume that  $\tilde{\gamma}'$  has a geodesic segment from some point  $iy$  to the point  $i$ , and that it then turns by an angle  $\theta < \epsilon$  and has a length 1 geodesic segment from  $i$  to some point  $b$ . Note that this is possible since we assume that  $l(\alpha), l(\beta) \geq 2$ . Then the segment from  $i$  to  $b$  lies on a circle with center  $a \in \mathbb{R}$  and radius  $r$ , where

$$a = \frac{1}{\tan \theta}, \quad r = \frac{1}{\sin \theta}$$

In particular,  $b = a + re^{i(\pi-\theta-\Delta\theta)}$  where  $\Delta\theta$  goes to zero as  $\theta$  goes to zero (left side of Figure 3).

FIGURE 3. We replace the subarc from  $iy$  to  $b$  by a subarc of a circle centered at  $A$ .

We will replace the subarc from  $iy$  to  $b$  by a smooth arc with the same initial and final tangent vectors. As long as the distance from  $iy$  to  $i$  is at most 1, the fact that  $l(\alpha), l(\beta) \geq 2$  means we can do this to both corners of  $\gamma'$  at the same time to get the nearby  $C^2$  curve  $\gamma_c$ .

For each  $\theta$ , there is a unique  $y = y(\theta)$  so that there is a Euclidean circle with center inside  $\mathbb{H}^2$  that is tangent to  $\tilde{\gamma}'$  at both  $iy$  and at  $b$ . Moreover, we can compute that as  $\theta$  goes to zero,  $y(\theta)$  approaches  $\frac{1}{2e-1}$ . So for all  $\epsilon$  small enough, the hyperbolic distance between  $iy$  and  $i$  is smaller than 1.

Suppose the circle has center at  $A$  and Euclidean radius  $\rho$ . If it meets the real axis at angle  $\phi$ , then its hyperbolic curvature is  $|\cos(\phi)|$  [GR85, Lemma 3]. So we see that its curvature is  $\frac{y}{\rho}$  at each point. By explicitly computing  $y$  and  $\rho$ , one can show that the curvature goes to zero as  $\theta$  (and  $\epsilon$ ) go to zero.

We replace each corner of  $\tilde{\gamma}'$  in this way. This gives us our piecewise  $C_2$  curve  $\tilde{\gamma}_c$  whose curvature at each point goes to zero uniformly as  $\epsilon$  goes to zero. Note that  $\tilde{\gamma}_c$  projects down to a piecewise  $C^2$  closed curve  $\gamma_c$  in  $S$ . By [Lei06], the distance from  $\tilde{\gamma}_c$  to  $\tilde{\gamma}$  is at most  $f(\epsilon)$ , where  $f(\epsilon)$  is a continuous function with  $f(0) = 0$ .

By construction, the distance from  $\tilde{\gamma}_c$  to  $\tilde{\gamma}'$  is at most  $\epsilon \sinh(1)$ , as the circle segment is contained in the geodesic triangle with vertices at  $iy, i$  and  $b$ . Thus, for

all  $\epsilon$  small enough, the distance from  $\tilde{\gamma}'$  to  $\tilde{\gamma}$  is at most

$$d(\tilde{\gamma}, \tilde{\gamma}') \leq \epsilon \sinh(\rho) + f(\epsilon)$$

In particular, this distance approaches 0 as  $\epsilon$  goes to 0. Note that if  $l(\beta) < 2$ , then the same argument implies that  $d(\tilde{\gamma}, \tilde{\gamma}') \leq \epsilon(\sinh(\rho) + \sinh(2)) + f(\epsilon)$ . This quantity also goes to zero with  $\epsilon$ . So in either case, there is a function  $D(\epsilon)$  so that  $\gamma$  must  $D(\epsilon)$  fellow travel  $\gamma'$ , with  $\lim_{\epsilon \rightarrow 0} D(\epsilon) = 0$ .

We now show the following: For any  $\epsilon > 0$  there is an  $R > 0$  so that for any geodesic arc  $\alpha$  there exists a geodesic arc  $\beta$  so that  $l(\beta) \leq R$  and we can form a closed curve  $\gamma' = \alpha \circ \beta$  with angle deficit at most  $\epsilon$  at each corner.

Take a unit speed parameterization  $\alpha : [0, L] \rightarrow \mathcal{S}$ . Let  $v = \alpha'(L)$  be its tangent vector in the unit tangent bundle  $T_1(\mathcal{S})$  of  $\mathcal{S}$ . Let  $f_t$  denote geodesic flow on  $T_1(\mathcal{S})$  for time  $t$ , and let  $r_\theta$  denote rotation by angle  $\theta \in [-\pi, \pi]$ .

We get coordinates in a small neighborhood about  $v$  by assigning each vector  $z$  the triple  $(\theta, t, \phi)$ , where  $z = r_\phi \cdot f_t \cdot r_\theta(v)$ . Let  $N(v, \epsilon)$  be the set of all vectors with coordinates  $(\theta, t, \phi)$  so that  $|\theta|, |\phi| < \frac{\epsilon}{2}$  and  $0 < t < \frac{1}{2} \text{inj}(X)$ , where  $\text{inj}(X)$  denotes the injectivity radius of  $X$ . (Figure 4).

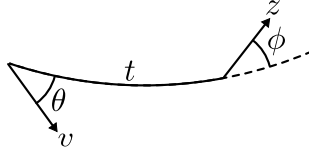


FIGURE 4. The set  $N(v, \epsilon)$  contains all vectors  $z$  with  $|\theta|, |\phi| < \frac{\epsilon}{2}$  and  $t < \frac{1}{2} \text{inj}(X)$ .

Set  $w = \alpha'(0) \in T_1(\mathcal{S})$ . Fix any  $\epsilon > 0$ . Since  $N(v, \epsilon)$  is a set of positive Lebesgue measure, the mixing of the geodesic flow implies that there is some  $z \in N(v, \epsilon)$  and  $T > 0$  so that

$$w \in N(f_T(z), \epsilon)$$

(as in Figure 5).

Then if  $z$  has coordinates  $(\theta, t, \phi)$  near  $v$  and  $w$  has coordinates  $(\theta', t', \phi')$  near  $f_T(z)$ , we can let  $\eta$  be the piecewise geodesic arc whose lift to  $T_1(\mathcal{S})$  is given by the parameterization

$$\eta'(t) = \begin{cases} f_s \cdot r_\theta(v) & \text{if } 0 \leq s < t \\ f_{s-t}(z) & \text{if } t \leq s < T+t \\ f_{s-t-T} \cdot r_{\theta'}(f_T(z)) & \text{if } T+t \leq s \leq T+t+t' \end{cases}$$

(Figure 5.) That is, we start at  $v$  and flow in the direction  $r_\theta(v)$  for time  $t$ . Then we flow in the direction of  $z$  for time  $T$ , and lastly, in the direction of  $r_{\theta'}(f_T(z))$  for time  $t'$ . Note that  $r_\theta(v)$  and  $z$  lie over the same point in  $\mathcal{S}$ , as do  $f_T(z)$  and  $r_{\theta'}(f_T(z))$ . Thus, this defines a closed curve in  $\mathcal{S}$ .

Let  $\beta$  be the arc freely homotopic to  $\eta$  relative its endpoints. Then  $|\theta|, |\theta'|, |\phi|, |\phi'| < \frac{\epsilon}{2}$  implies that the angle deficit at each of the two points where  $\alpha$  meets  $\beta$  is at most  $\epsilon$ . By the triangle inequality,  $l(\beta) \leq T + 2$ .

In fact, there is a continuous function  $T(\cdot, \cdot, \cdot) : T_1(\mathcal{S}) \times T_1(\mathcal{S}) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  so that  $T \leq T(v, w, \epsilon)$ . Moreover, because  $T_1(\mathcal{S})$  is compact, there is a continuous

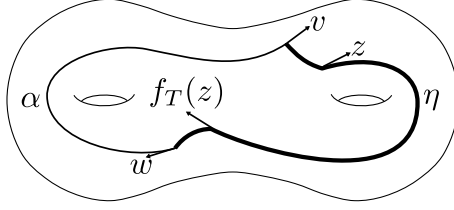


FIGURE 5.

function  $T : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  so that

$$T(v, w, \epsilon) \leq T(\epsilon)$$

This is the only place where we use that  $\mathcal{S}$  is compact. In particular, for any  $\alpha$  and  $\epsilon$ , there is an arc  $\beta$  of length at most  $R(\epsilon) = T(\epsilon) + 2$  so that we can concatenate  $\alpha$  and  $\beta$  into a closed curve with angle deficit at most  $\epsilon$  at its corners.

Choose  $\epsilon$  small enough so that  $D(\epsilon) \leq 1$ , where  $D(x)$  is the function we defined in the first part of this proof. Let  $R = R(\epsilon)$ . Then for every arc  $\alpha$ , there is a closed geodesic  $\gamma$  so that

$$l(\gamma) \leq l(\alpha) + R$$

and  $\gamma$  1-fellow travels  $\alpha$ . □

The previous claim says we can approximate any arc  $\alpha$  by a closed geodesic  $\gamma$  of roughly the same length. The next claim allows us to estimate the self-intersection number of  $\gamma$ .

**Claim 2.4.** *If  $\alpha$  and  $\beta$  are two geodesic arcs with  $l(\alpha) \leq L_\alpha$  and  $l(\beta) \leq L_\beta$ , then*

$$\#\alpha \cap \beta \leq \kappa L_\alpha L_\beta$$

*where we require  $L_\alpha, L_\beta \geq 1$ , and  $\kappa$  is a constant that depends only on  $X$ .*

*Proof.* The proof is almost exactly the same as the proof of [Bas13, Theorem 1.1], where Basmajian proves this result in the case where  $\alpha$  and  $\beta$  are the same (non-simple) closed geodesic. We recreate it here for completeness.

Take a pants decomposition  $\Pi$  of  $\mathcal{S}$ . Further cut each pair of pants into two congruent right-angled hexagons. So each hexagon has boundary edges that lie on curves in  $\Pi$ , and seam edges that join curves in  $\Pi$ .

The hexagon decomposition cuts the arc  $\alpha$  into segments, which are maximal subarcs that lie in a single hexagon, and the same is true for  $\beta$ . Note that if a hexagon  $h$  has  $n$   $\alpha$ -segments and  $m$   $\beta$ -segments, then the total number of intersections between  $\alpha$  and  $\beta$  in  $h$  is at most  $nm$ . This is because each hexagon is simply connected and convex, so any pair of segments intersects at most once. Therefore, if  $\alpha$  has  $N_\alpha$  total segments, and if  $\beta$  has  $N_\beta$  total segments, then

$$\#\alpha \cap \beta \leq N_\alpha N_\beta$$

So we just need to bound  $N_\alpha$  and  $N_\beta$  in terms of  $l(\alpha)$  and  $l(\beta)$ , respectively. We will say a **full segment** is any segment of  $\alpha$  or  $\beta$  that does not contain an endpoint of that arc. Take three consecutive full segments  $\sigma_1, \sigma_2, \sigma_3$  of  $\alpha$ . Then by the argument in [Bas13, Step 2, Section 3], we have that

$$l(\sigma_1) + l(\sigma_2) + l(\sigma_3) \geq C$$



where  $C$  depends only on the metric  $X$ . The total length of all full segments of  $\alpha$  is at most  $l(\alpha)$ . Thus, its number of full segments is at most  $\frac{3l(\alpha)}{C}$ . The arc  $\alpha$  only has two segments that are not full: they are the ones that contain its endpoints. So we have

$$N_\alpha \leq 2 + \frac{3l(\alpha)}{C}$$

and likewise,

$$N_\beta \leq 2 + \frac{3l(\beta)}{C}$$

Therefore,

$$\#\alpha \cap \beta \leq \left(2 + \frac{3l(\alpha)}{C}\right) \left(2 + \frac{3l(\beta)}{C}\right)$$

If  $l(\alpha) \leq L_\alpha$  and  $l(\beta) \leq L_\beta$ , and if we assume  $L_\alpha, L_\beta \geq 1$ , we have that

$$\#\alpha \cap \beta \leq \left(2 + \frac{3}{C}\right)^2 L_\alpha L_\beta$$

Thus, the claim holds for  $\kappa = (2 + \frac{3}{C})^2$ , which is a constant depending only on  $X$ .  $\square$

*Proof of Lemma 2.2.* Assume that  $l(\alpha) > 3$ . Then Claim 2.3 says there is a closed geodesic  $\gamma$  that 1-fellow travels  $\alpha$ . By construction,  $\gamma$  is freely homotopic to a concatenation  $\alpha \circ \beta$ , where  $\beta$  is a geodesic arc of length at most  $R$ , for  $R$  depending only on the metric  $X$ . So Claim 2.4 allows us to estimate the self-intersection number of  $\gamma$ .

Assuming  $R \geq 1$ , Claim 2.4 implies that

$$\#\alpha \cap \beta \leq \kappa R l(\alpha)$$

and

$$\#\beta \cap \beta \leq \kappa R^2$$

Thus, if we assume  $l(\alpha) < L$ ,

$$\begin{aligned} i(\gamma, \gamma) &\leq i(\alpha, \alpha) + i(\alpha, \beta) + i(\beta, \beta) \\ &\leq i(\alpha, \alpha) + \kappa R l(\alpha) + \kappa R^2 \\ &\leq K + \kappa R(L + R) \end{aligned}$$

So if  $L \geq R$ , then  $i(\gamma, \gamma) \leq K + 2\kappa RL$ . Setting  $d = 2\kappa R$ , we get Lemma 2.2.  $\square$

### 3. COVERING BY NEIGHBORHOODS OF CLOSED CURVES

The proofs of Theorems 1.1 and 1.2 come from covering our sets of complete geodesics with neighborhoods of closed geodesics. For any function  $f(l)$ , we give an infinite collection of finite open covers  $\{\mathcal{C}_n(f) = C_n\}$ . Then for any  $L$ ,  $\mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f, L)$  for all  $n$  large enough. Moreover, the union of any infinite subsequence over these covers gives a cover of  $\mathbf{Im} \mathcal{G}(f)$ .

Specifically, for each  $\gamma \in \mathcal{G}^c$ , let  $N_\epsilon(\gamma)$  be an  $\epsilon$ -neighborhood of  $\mathbf{Im} \gamma$ . Then if  $\mathcal{H} \subset \mathcal{G}^c$  is any collection of closed geodesics, we let

$$N_\epsilon(\mathcal{H}) = \bigcup_{\gamma \in \mathcal{H}} N_\epsilon(\gamma)$$

We define the cover  $\mathcal{C}_n$  by

$$\mathcal{C}_n = N_{\epsilon(n)}(\mathcal{G}_n^c(c_X \cdot f(n)))$$

where  $\epsilon(n) = 2e^{-n/4}$ , and  $c_X = 2 + d/2$ , for the constant  $d$  defined in Lemma 2.2. That is,  $\mathcal{C}_n$  is a finite collection of  $\epsilon(n)$ -neighborhoods of closed geodesics of length at most  $n$ , with at most  $c_X f(n)$  self-intersections.

**3.1. Finite covers.** Recall that

$$\mathcal{G}(f, L) = \{\gamma \in \mathcal{G} \mid \#\gamma|_{[a, a+l]} \cap \gamma|_{[a, a+l]} \leq f(l), \forall l \geq L\}$$

In other words, this is the set of complete geodesics  $\gamma$  so that all length  $l$  subarcs have self-intersection number at most  $f(l)$ , for all  $l \geq L$ . Because we impose this regularity on the self-intersection function of geodesics in  $\mathbf{Im} \mathcal{G}(f, L)$ , we can show that each  $\mathcal{C}_n$  is a cover of  $\mathcal{G}(f, L)$ , as long as  $n$  is large enough.

Observe that if  $f(x) \leq g(x)$ , then  $\mathcal{G}(f) \subset \mathcal{G}(g)$ , and in fact,  $\mathcal{G}(f, L) \subset \mathcal{G}(g, L)$  for all  $L$ . So in this section, we assume without loss of generality that  $f(l) \geq l$  for all  $l$ .

**Lemma 3.1.** *Suppose  $f(l) \geq l$  for all  $l$ . Then for each  $L$ , there is an  $N > 0$  so that for all  $n \geq N$ ,  $\mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f, L)$ .*

*Proof.* Let  $x \in \mathbf{Im} \mathcal{G}(f, L)$ . Then there is a  $\gamma \in \mathcal{G}(f, L)$  parameterized by  $\gamma : \mathbb{R} \rightarrow \mathcal{S}$  so that  $x = \gamma(t_x)$  for some time  $t_x \in \mathbb{R}$ . Choose  $l \geq L$ . Let  $\alpha_{x,l} = \gamma|_{[t_x - \frac{l}{2}, t_x + \frac{l}{2}]}$  be the length  $l$  subarc of  $\gamma$  centered at  $x$ . Then because  $\gamma \in \mathcal{G}(f, L)$ ,

$$\#\alpha_{x,l} \cap \alpha_{x,l} \leq f(l)$$

By Lemma 2.2, as long as  $l \geq d$ , there is a closed geodesic  $\delta \in \mathcal{G}_{2l}^c(f(l) + dl)$  that 1-fellow travels  $\alpha_{x,l}$ , where  $d$  is a constant depending only on  $x$ . Since  $f(l) \geq l$ , we have, in fact, that  $\delta \in \mathcal{G}_{2l}^c(c_X f(l))$ , where  $c_X = 1 + d$ .

Lift  $\delta$  and  $\alpha_{x,l}$  to curves  $\tilde{\delta}$  and  $\tilde{\alpha}_{x,l}$  in the universal cover such that the endpoints of  $\tilde{\alpha}_{x,l}$  are at most distance 1 away from  $\tilde{\delta}$ . The midpoint of  $\tilde{\alpha}_{x,l}$  is a lift  $\tilde{x}$  of  $x$ . Thus,

$$d(\tilde{x}, \tilde{\delta}) < 2e^{-l/2}$$

To see this, drop perpendiculars from  $\tilde{x}$  and from an endpoint of  $\tilde{\alpha}_{x,l}$  down to  $\tilde{\delta}$ . This forms a quadrilateral with 3 right angles (called a Lambert quadrilateral.) By properties of Lambert quadrilaterals,  $d(\tilde{x}, \tilde{\delta}) \leq \sinh(1)/\cosh(\frac{l}{2})$ . Since  $\sinh(1) < 2$  and  $\cosh(\frac{l}{2}) > e^{l/2}$ , we have our inequality (Figure 6).

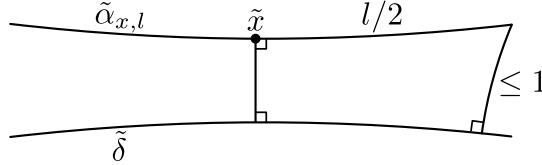


FIGURE 6.

Let  $N = \max\{2L, d\}$ . Then for each  $n \geq N$ , and for each  $x \in \mathbf{Im} \mathcal{G}(f, L)$ , there is a  $\delta \in \mathcal{G}_n^c(c_X f(n))$  so that  $x \in N_{\epsilon(n)}(\delta)$  for  $\epsilon(n) = 2e^{-n/4}$ . In other words, for all  $n \geq N$ ,  $\mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f, L)$ .  $\square$

**3.2. Infinite covers.** To cover  $\mathbf{Im} \mathcal{G}(f)$ , we need to take the union of infinitely many covers in the sequence  $\{\mathcal{C}_n\}$ . Once again, observe that if  $f(l) \leq g(l)$  for two functions  $f$  and  $g$ , then  $\mathcal{G}(f) \subset \mathcal{G}(g)$ . So we can assume without loss of generality that  $f(l) \geq l$  for all  $l$ , and that  $f(l)$  is increasing in  $l$ .

**Lemma 3.2.** *Suppose  $f(l)$  is an increasing function so that  $f(l) \geq l$ . Then for each  $N > 0$ ,  $\bigcup_{n=N}^{\infty} \mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f)$ .*

*Proof.* Let  $x \in \mathbf{Im} \mathcal{G}(f)$ . Then there is a  $\gamma \in \mathcal{G}(f)$  parameterized by  $\gamma : \mathbb{R} \rightarrow \mathcal{S}$  so that  $x = \gamma(t_x)$  for some time  $t_x \in \mathbb{R}$ .

Recall that  $\gamma_l = \gamma|_{[-\frac{l}{2}, \frac{l}{2}]}$  is the length  $l$  subarc of  $\gamma$  centered at  $\gamma(0)$ . Because  $\gamma \in \mathcal{G}(f)$ , there is some length  $l_0$  depending on  $\gamma$  so that for all  $l \geq l_0$ ,

$$\#\gamma_l \cap \gamma_l \leq 2f(l)$$

For each  $l$ , let  $\alpha_{x,l} = \gamma|_{[t_x - \frac{l}{2}, t_x + \frac{l}{2}]}$  be the length  $l$  subarc of  $\gamma$  centered at  $x$ . Then  $\alpha_{x,l} \subset \gamma_{l+t_x}$  for each  $l$ .

Let  $l \geq \max\{l_0, t_x, d\}$ , where  $d$  is the constant from Lemma 2.2. Then  $l + t_x \leq 2l$ . Since we assume that  $f(l)$  is increasing, this means that  $f(l + t_x) \leq f(2l)$ . Thus,

$$\#\alpha_{x,l} \cap \alpha_{x,l} \leq 2f(2l)$$

By the same argument as in the proof of Lemma 3.1, there is a closed geodesic  $\delta \in \mathcal{G}_{2l}^c(2f(2l) + dl)$  so that

$$d(x, \delta) \leq 2e^{-l/2}$$

Since we assume  $f(l) \geq l$ , we have, in fact, that  $\delta \in \mathcal{G}_{2l}^c(c_X f(2l))$ , where  $c_X = 2 + d/2$ .

Then for each  $x$ , and for each  $n \geq \frac{1}{2} \max\{l_0, t_x, d\}$ , there is a  $\delta \in \mathcal{G}_n^c(c_X f(n))$  so that  $x \in N_{\epsilon(n)}(\delta)$  for  $\epsilon(n) = 2e^{-n/4}$ . In other words,  $\bigcup_{n \geq N} \mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f)$  for each  $N$ .  $\square$

#### 4. COUNTING THE APPROXIMATING CLOSED CURVES

We will define the Lebesgue (or Hausdorff) measure of any cover  $\mathcal{C}_n$  to be the measure of the union of the open sets in  $\mathcal{C}_n$ . We will eventually wish to show that the measures of these covers go to zero as  $n$  goes to infinity. Recall that we defined each cover  $\mathcal{C}_n$  as the collection of open neighborhoods about geodesics in  $\mathcal{G}_n^c(c_X f(n))$ . So to show that these measure go to zero, we need to bound the number of closed geodesics in these sets.

**Lemma 4.1.** *If  $f(n) \leq (kn)^2$ , then*

$$\#\mathcal{G}_n^c(c_X f(n)) = o\left(\frac{1}{n} e^{\mu(k)n}\right)$$

where  $\lim_{k \rightarrow 0} \mu(k) = 0$ , and  $c_X$  is the constant depending only on  $X$  defined in Section 3.

In fact, we will show that  $\mu(k) = a_X k \log(a_X(1 + \frac{1}{k}))$ , where  $a_X$  depends only on  $X$ . First we will bound  $\#\mathcal{G}_L^c(K)$  for any  $L$  and  $K$ , and then we will set  $L = n$  and consider the case where  $K(n) \leq (kn)^2$ .

**Claim 4.2.** *Let  $\mathcal{S}$  be a closed, genus  $g$  surface. For any  $L$  and  $K$ , we have*

$$\#\mathcal{G}_L^c(K) \leq p(L) \left( a_X \frac{L}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}$$

where  $p(L)$  is a polynomial in  $L$ , and  $a_X$ , as well as the coefficients of  $p(L)$ , depend only on the metric  $X$ .

In fact, one can look carefully at the argument in [Mir16, Lemma 5.6] to see that  $p(L)$  can be replaced by a polynomial in  $K$  times  $L^{6g-6}$ . However, the formula in Lemma 4.1 is easier to prove, and suffices for Lemma 4.1.

*Proof.* Let  $\text{Mod}_{\mathcal{S}}$  denote the mapping class group of  $\mathcal{S}$ . Then  $\text{Mod}_{\mathcal{S}}$  acts on  $\mathcal{G}^c$ , preserving self-intersection number. For each  $\gamma \in \mathcal{G}^c$  let  $\text{Mod}_{\mathcal{S}} \cdot \gamma$  denote its orbit. Let  $\mathcal{O}(\cdot, K)$  be the set of orbits of curves with at most  $K$  self-intersections:

$$\mathcal{O}(\cdot, K) = \{\text{Mod}_{\mathcal{S}} \cdot \gamma \mid i(\gamma, \gamma) \leq K\}$$

If  $\gamma$  has  $K$  self-intersections, then the shortest curve in  $\text{Mod}_{\mathcal{S}} \cdot \gamma$  has length between  $c_1 \sqrt{K}$  and  $c_2 K$  for some constants  $c_1$  and  $c_2$ . In fact, there exist such constants, depending only on  $X$ , for which these bounds are tight [AGPS16, Bas13, Gas16]. Thus, if  $L$  is small enough, then not all  $\text{Mod}_{\mathcal{S}}$  orbits contain curves of length at most  $L$ . So we let

$$\mathcal{O}(L, K) = \{\text{Mod}_{\mathcal{S}} \cdot \gamma \mid \text{Mod}_{\mathcal{S}} \cdot \gamma \cap \mathcal{G}_L^c(K) \neq \emptyset\}$$

be those orbits that contain curves of length at most  $L$ .

In [Sap16b], we show that

$$\#\mathcal{O}(L, K) \leq \left( a_X \frac{L}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}$$

for a constant  $a_X$  depending only on  $X$ .

Since we have a bound on the number of orbits, we just need a bound on the number of curves in each orbit. For each  $\gamma$ , let

$$s(L, \gamma) = \{\gamma' \in \text{Mod}_{\mathcal{S}} \cdot \gamma \mid l(\gamma') \leq L\}$$

In [Mir16], Mirzakhani shows that  $s(L, \gamma)$  grows asymptotically like  $a_{\gamma, X} L^{6g-6}$ , where the constant  $a_{\gamma, X}$  depends on the  $\text{Mod}_g$  orbit of  $\gamma$ , and on  $X$ . The dependence of  $a_{\gamma, X}$  on  $\gamma$  is difficult to determine. As mentioned above, a careful analysis of [Mir16, Lemma 5.6] should imply that  $s(L, \gamma) \leq p(K) L^{6g-6}$ , where  $p(K)$  is a polynomial in  $K$  whose coefficients depend only on the metric  $X$ . However, there is a faster way to see that

$$\#s(L, \gamma) \preccurlyeq L^{30g-12}$$

where the constant depends only on  $X$ .

First, suppose  $\gamma$  is a filling curve on a closed, genus  $g$  surface. Then the proof of [Sap16a, Lemma 2.2] implies that

$$\#s(L, \gamma) \preccurlyeq L^{6g-6}$$

where the constant depends only on  $X$ .

Now suppose  $\gamma$  only fills a proper subsurface  $T \subset \mathcal{S}$ . Then by [Sap16a, Proposition 2.7],  $l(\partial T) \leq 2l(\gamma)$ . So for any  $g \in \text{Mod}_{\mathcal{S}}$  with  $l(g \cdot \gamma) \leq L$  we have  $l(g \cdot \partial T) \leq 2L$ . The number of simple closed curves of length at most  $2L$  on  $\mathcal{S}$  is at most  $b_X (2L)^{6g-6}$ , where  $b_X$  is a constant depending only on  $X$ .

So we can now fix a subsurface  $T$  of  $\mathcal{S}$ , and count all curves in  $\text{Mod}_{\mathcal{S}} \cdot \gamma$  that fill  $T$ :

$$s(L, \gamma, T) = \{\gamma' \in \text{Mod}_{\mathcal{S}} \cdot \gamma \mid l(\gamma') \leq L, \gamma' \text{ fills } T\}$$

Double  $T$  across its boundary. This gives a new surface  $Q = T \cup \bar{T}$ , where  $\bar{T}$  is the complement of  $T$  in  $Q$ . The metric on  $Q$  is obtained by doubling the metric on  $T$ . Moreover, if  $\gamma' \in \text{Mod}_{\mathcal{S}} \cdot \gamma$  lies in  $T$ , then it has a mirror image  $\bar{\gamma}'$  that lies in  $\bar{T}$ . Finally, for each component  $\alpha$  of  $\partial T$ , there is a curve  $\beta$  that intersects  $\alpha$  minimally with  $l(\beta) \preccurlyeq l(\gamma')$ , where the constant depends only on  $X$  ([Sap16a, Proposition 2.7]). Let  $\eta$  be the union of all such curves  $\beta$  together with  $\partial T$ . Then consider the curve

$$\delta = \gamma' \cup \bar{\gamma}' \cup \eta$$

By construction,  $\delta$  fills the closed surface  $Q$ . If  $\mathcal{S}$  had genus  $g$ , then the genus of  $Q$  is at most  $4g$ . (Really, the genus of  $Q$  is at most  $4g - 1$ , but this slight improvement in the upper bound leads to messier formulae, which are still not tight.)

So we can count curves in  $s(L, \gamma, T)$  on  $\mathcal{S}$  by counting curves in  $s(L, \delta)$  on  $Q$ , instead. In fact, since  $\delta$  fills a closed surface, we get

$$s(L, \gamma, T) \leq s(L, \delta) \preccurlyeq L^{24g-6}$$

Thus,

$$s(L, \gamma) \preccurlyeq L^{6g-6} \cdot L^{24g-6} = L^{30g-12}$$

for constants that depends only on the metric  $X$ .

Combined with the orbit counting result, we get that

$$\#\mathcal{G}_L^c(K) \leq p(L) \left( a_X \frac{L}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}$$

where  $p(L)$  is a polynomial in  $L$  of degree  $30g - 12$ , whose constants depend only on  $X$ , and  $a_X$  depends only on  $X$ .  $\square$

*Proof of Lemma 4.1.* Now set  $L = n$  and suppose  $K \leq (kn)^2$  for some  $k$ . Then we are ready to show that

$$\#\mathcal{G}_n^c(c_X f(n)) = o\left(\frac{1}{n} e^{\mu(k) \cdot n}\right)$$

where

$$\mu(k) = \frac{1}{2} a_X k \ln\left(\frac{a_X}{k} + a_X\right)$$

Because  $p(n) = o\left(\frac{1}{n} e^{\frac{1}{2} \mu(k) n}\right)$  for any polynomial  $p(n)$  and any positive  $\mu(k)$ , we only need to find a function  $\mu(k)$  so that

$$\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}} = o\left(e^{\frac{1}{2} \mu(k) n}\right)$$

whenever  $K \leq (kn)^2$ .

For any fixed  $n \geq 1$ , we have that  $\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}}$  is an increasing function in  $K$ , since we may assume that  $a_X \geq e$ . Since we assume that  $K \leq (kn)^2$ , this means

$$\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}} \leq \left( a_X \frac{1}{k} + a_X \right)^{a_X \cdot kn} = e^{n \cdot a_X k \ln\left(\frac{a_X}{k} + a_X\right)}$$

In other words, if  $K = K(n) \leq (kn)^2$ , then

$$\left( a_X \frac{n}{\sqrt{K}} + a_X \right)^{a_X \sqrt{K}} = o\left(e^{1/2 \mu(k) \cdot n}\right)$$

for  $\mu(k) = 4a_X k \ln(\frac{a_X}{k} + a_X)$ . Lastly, note that  $\lim_{k \rightarrow 0} \mu(k) = 0$ .  $\square$

## 5. NOWHERE DENSITY

A set  $U \subset \mathcal{S}$  is nowhere dense if its closure has an empty interior. In particular,  $U$  is nowhere dense if, for any open ball  $\mathcal{B}$ ,  $\mathcal{B} \setminus U$  contains a non-empty open set.

We will show this is the case for  $\mathbf{Im} \mathcal{G}(f, L)$ . In particular, Lemma 3.1 gives us a family  $\{\mathcal{C}_n\}$  of covers of  $\mathbf{Im} \mathcal{G}(f, L)$ , where each  $\mathcal{C}_n$  is a finite collection of regular neighborhoods about closed geodesics. Below we show that these covers have arbitrarily small Lebesgue measure. (The Lebesgue measure of  $\mathcal{C}_n$  is defined to be the measure of the union of all elements of  $\mathcal{C}_n$ .) This will imply that  $\mathbf{Im} \mathcal{G}(f, L)$  is nowhere dense.

*Proof of Theorem 1.2.* By Lemma 3.1, there is an  $N$  depending only on  $L$  so that for all  $n \geq N$ ,  $\mathcal{C}_n$  is a cover of  $\mathbf{Im} \mathcal{G}(f, L)$ . We wish to estimate the Lebesgue measure of  $\mathcal{C}_n$ . Recall that  $\mathcal{C}_n$  is the set of  $\epsilon(n)$ -neighborhoods of the closed geodesics in  $\mathcal{G}_n^c(c_X f(n))$ , where  $\epsilon(n) = 2e^{-n/4}$ .

Let  $\lambda(A)$  denote the Lebesgue measure of any subset  $A \subset \mathcal{S}$ . If  $\gamma \in \mathcal{G}_n^c(c_X f(n))$ , then  $l(\gamma) \leq n$ . So for all  $\epsilon(n)$  small enough, the measure of  $N_{\epsilon(n)}(\gamma)$  is bounded above by

$$\lambda(N_{\epsilon(n)}(\gamma)) \leq 5ne^{-n/4}$$

By Lemma 4.1, if  $f(n) \leq (kn)^2$ , then  $\#\mathcal{G}_n^c(c_X f(n)) = o(\frac{1}{n}e^{\mu(k)n})$ . So

$$\lambda(\mathcal{C}_n) = o\left(e^{(\mu(k) - \frac{1}{4})n}\right)$$

We have that  $\lim_{k \rightarrow 0} \mu(k) = 0$ . So there is some  $k_0$  so that for all  $k < k_0$ ,  $\mu(k) < \frac{1}{4}$ . Then for all  $k \leq k_0$ ,

$$\lim_{n \rightarrow \infty} \lambda(\mathcal{C}_n) = 0$$

Suppose  $k \leq k_0$ . Choose any open ball  $\mathcal{B} \subset \mathcal{S}$ . Choose  $n$  so that  $\lambda(\mathcal{C}_n) < \frac{1}{2}\lambda(\mathcal{B})$ . Then  $\mathcal{B}$  is crossed by finitely many elements of  $\mathcal{C}_n$ . The elements of  $\mathcal{C}_n$  are regular neighborhoods of closed geodesics, so our choice of  $n$  guarantees that  $\mathcal{B} \setminus \mathcal{C}_n$  has non-empty interior. But  $\mathcal{C}_n$  is an open cover of  $\mathbf{Im} \mathcal{G}(f, L)$ . So  $\mathcal{B} \setminus \mathbf{Im} \mathcal{G}(f, L)$  has non-empty interior, as well. Therefore,  $\mathbf{Im} \mathcal{G}(f, L)$  is nowhere dense for all  $L$  and all functions  $f$  with  $f(l) \leq (kl)^2$ .  $\square$

## 6. HAUSDORFF DIMENSION

*Proof of Theorem 1.1.* The Hausdorff dimension of a set is defined as follows. Given a subset  $G$  of a metric space  $X$ , let  $\mathcal{C} = \{\mathcal{B}(x_i, r_i)\}$  be a countable cover of  $G$  by metric balls centered at  $x_i$  and of radius  $r_i$ , for each  $i$ . We define the  $h$ -dimensional Hausdorff measure of  $\mathcal{C}$  to be  $\nu_h(\mathcal{C}) = \sum r_i^h$ . The  $h$ -dimensional Hausdorff measure of a set  $G$  is defined as

$$\nu_h(G) = \inf_{\mathcal{C}} \nu_h(\mathcal{C})$$

where the infimum is taken over all such covers of  $G$ . Then the Hausdorff dimension of  $G$  is defined to be

$$\dim_H(G) = \inf\{h \mid \nu_h(G) = 0\}$$

By Lemma 3.2, infinite unions of the covers  $\mathcal{C}_1, \dots, \mathcal{C}_n, \dots$  cover  $\mathbf{Im} \mathcal{G}(f)$ . These are covers by regular neighborhoods of closed geodesics, but we can use them to build covers of  $\mathbf{Im} \mathcal{G}(f)$  by metric balls. In fact, for each  $n$ , we can build new cover  $\mathcal{C}_n^H$ , which is a collection of balls whose union contains the union of open sets in  $\mathcal{C}_n$ . Note that for all  $\epsilon(n)$  small enough, we can cover the  $\epsilon(n)$ -regular neighborhood of any  $\gamma \in \mathcal{G}_n^c(c_X f(n))$  by  $2\frac{n}{\epsilon(n)}$  balls of radius  $2\epsilon(n)$ . So let  $\mathcal{C}_n^H$  be the union of all these balls for each open set in  $\mathcal{C}_n$ . We will use the collection  $\{\mathcal{C}_n^H\}$  of these covers to bound the Hausdorff dimension of  $\mathbf{Im} \mathcal{G}(f)$ .

Then Lemmas 3.2 and 4.1 allow us to estimate the Hausdorff  $h$ -volume of  $\mathbf{Im} \mathcal{G}(f)$  by estimating the volume of  $\mathcal{C}_n^H$ . (The volume of a cover is defined to be the volume of the union of all elements of the cover.)

Let  $\mathcal{C}_n^H$  be the collection of metric balls defined above. First, we find a condition on  $h$  so that

$$\lim_{n \rightarrow \infty} \nu_h(\mathcal{C}_n^H) = 0$$

Each ball in  $\mathcal{C}_n^H$  has radius  $2\epsilon(n) = 4e^{-\frac{n}{4}}$ . Each closed geodesic  $\gamma \in \mathcal{G}_n^c(c_X f(n))$  has length  $n$ , so it contributes  $\frac{2n}{\epsilon(n)} = ne^{n/4}$  balls to the cover. So the total Hausdorff  $h$ -volume of  $\mathcal{C}_n^H$  is bounded above by

$$\begin{aligned} \nu_h(\mathcal{C}_n^H) &= \sum_{\mathcal{G}_n^c(c_X f(n))} ne^{\frac{n}{4}} (4e^{-\frac{n}{4}})^h \\ &= \sum_{\mathcal{G}_n^c(c_X f(n))} 4^h ne^{\frac{n}{4}(h-1)} \end{aligned}$$

If  $f(n) \leq (kn)^2$ , then by Lemma 4.1, the number of these closed geodesics in  $\mathcal{G}_n^c(c_X f(n))$  grows like  $o\left(\frac{1}{n}e^{\mu(k)n}\right)$ , where  $\lim_{n \rightarrow 0} \mu(k) = 0$ . Since  $4^h$  is a constant for each  $h$ ,

$$\nu_h(\mathcal{C}_n^H) = o\left(e^{n(\mu(k) - \frac{h-1}{4})}\right)$$

In particular,  $\lim_{n \rightarrow \infty} \nu_h(\mathcal{C}_n^H) = 0$  whenever  $h > 4\mu(k) + 1$ .

Suppose  $h > 4\mu(k) + 1$ . Then there is a subsequence  $\{n_i\}$  so that  $\nu_h(\mathcal{C}_{n_i}^H) \leq 2^{-i}$  for each  $i$ . By Lemma 3.2, any infinite subsequence of  $\{\mathcal{C}_n^H\}$  covers  $\mathbf{Im} \mathcal{G}(f)$ . In particular, for any  $N \geq 0$ ,  $\cup_{i=N}^{\infty} \mathcal{C}_{n_i}^H$  covers  $\mathbf{Im} \mathcal{G}(f)$ . Thus, whenever  $h > 4\mu(k) + 1$ ,

$$\nu_h(\mathbf{Im} \mathcal{G}(f)) \leq 2^{-N}$$

for any  $N$ . In other words, the Hausdorff dimension of  $\mathbf{Im} \mathcal{G}(f)$  is at most  $4\mu(k) + 1$ .

Furthermore, suppose  $f(l) = o(l^2)$ . Then  $\mathcal{G}(f) \subset \cap_{k=1}^{\infty} \mathcal{G}(f_k)$ , for  $f_k = kl^2$ . So in this case, the Hausdorff dimension of  $\mathbf{Im} \mathcal{G}(f)$  is 1.  $\square$

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